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Symmetry classes of Fokker–Planck-type equations

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Abstract. We have performed a classification of the Fokker-Planck-type equations according to the maximal symmetry groups that keep the equations invariant. It is found that there are only four classes of such equations and the relations between the transition probabilities' of the equations have been obtained. We have also obtained complete functional bases of the invariants of the corresponding four groups.

1. Introduction

The Fokker-Planck equation and its generalisations occupy a central position in statistical physics (van Kampen 1978, Kubo *et al* 1985). It is the master equation for the Ornstein-Uhlenbeck process, which with a proper rescaling is the only stationary Gaussian Markov process. The Fokker-Planck equation derived from the Chapman-Kolmogorov equation is a Markovian process. On the physical side this equation results (Kubo *et al* 1985) when the basic random force in a statistical system is assumed to be Gaussian with a white spectrum. It has wide-ranging applications in physical, biological and sociological phenomena.

Bluman (1974) and Bluman and Cole (1969) made a detailed analysis of a special case of the Fokker-Planck equation. Bluman (1980) also showed that every one-dimensional Fokker-Planck equation with a six-parameter group of Lie symmetry can be transformed to a diffusion equation of heat. Recently Sastry and Dunn (1985), Sastry *et al* (1987) and Cicogna and Vitali (1989) have used Lie's method of the extended group (Ovsjannikov 1962, Hill 1982) to investigate the symmetry structure of some interesting cases of Fokker-Planck-type equations.

Here we have attacked the problem of classifying Fokker-Planck-type equations in one space and one time variable according to their symmetry groups. This type of equation has in general three 'transition probabilities'. In a self-consistent analysis we have shown that other than the trivial group consisting of time translation and scaling of the dependent variable, there can be only four other underlying symmetry groups. The relations between the three 'transition probabilities' in these four cases have been obtained. Finally, we have also obtained in all these cases the complete sets of invariant bases in terms of which any invariant of the symmetry groups can be expressed functionally.

2. Fokker–Planck-type equations

The evolution equation for a dynamic variable q in a stochastic process is transformed (Kubo *et al* 1985) into a equation for the probability distribution function $P(q, t)$, the

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Chapman–Kolmogorov equation:

$$\partial P(q,t)/\partial t = -P(q,t)\gamma(q) + \int P(q',t)(q'|w|q)\,dq'$$
$$\gamma(q) = \int (q|w|q')\,dq' \qquad (q'|w|q) \geq 0. \tag{1}$$

The functions $(q'|w|q)$ describe the rate of transitions from state q' to state q. This integro-differential equation can be transformed in an infinite-order differential equation

$$\partial P(q,t)/\partial t = \sum_{k=0}^{\infty} \beta^{(k)}(q)\partial^k P(q,t)/\partial q^k \tag{2}$$

where

$$\beta^{(k)}(q) = \int dr\, r^k (q-r|w|q) - \delta_{k0} \int dr (q|w|r+q)$$

will be called the 'transition probabilities' of the equation. If the summation on the right-hand side of (2) is restricted to $k \leq 2$, we shall call the resulting differential equation

$$\Delta \equiv \partial P/\partial t - \beta^{(0)}(q)P - \beta^{(1)}(q)\partial P/\partial q - \beta^{(2)}(q)\partial^2 P/\partial q^2 = 0 \qquad \beta^{(2)}(q) \neq 0 \tag{3}$$

the Fokker–Planck-type equation. The diffusion equation and other interesting equations of different physical and biological systems are special cases of this type. The generators of the maximal symmetry group of (3) are written in the form

$$X = \xi^t(q,t,P)\partial/\partial t + \xi^q(q,t,P)\partial/\partial q + \varphi(q,t,P)\partial/\partial P \tag{4}$$

and the velocity vectors ξ^t, ξ^q and φ are obtained (Ovsjannikov 1962, Hill 1982) by applying the second extension $X^{(2)}$ of X on (3) and separately equating to zero the coefficients of different monomials of the derivatives of P, after having replaced all time derivatives of P by space derivatives according to (3). This procedure gives us a set of partial differential equations:

$$\xi^t_P = \xi^t_q = \xi^q_P = 0 \qquad \varphi_{PP} = 0 \qquad \xi^q_q - \xi^q \beta^{(2)}(q)'/2\beta^{(2)}(q) = \tfrac{1}{2}\xi^t_t$$
$$\xi^q_{qq} + \xi^q_q \beta^{(1)}(q)/\beta^{(2)}(q) - \xi^q \beta^{(1)}(q)'/\beta^{(2)}(q) - \xi^q_t/\beta^{(2)}(q) = 2\varphi_{qP} + \xi^t_t \beta^{(1)}(q)/\beta^{(2)}(q) \tag{5}$$
$$\varphi_t - \beta^{(0)}(q)\varphi - \beta^{(1)}(q)\varphi_q - \beta^{(2)}(q)\varphi_{qq} = \xi^q P\beta^{(0)}(q)' + \xi^t_t P\beta^{(0)}(q) - \varphi_P P\beta^{(0)}(q).$$

Here the subscripts in ξ and φ denote the corresponding partial derivatives and the primes on β denote derivative with respect to q. The first four relations in (5) give us

$$\xi^t \equiv \xi^t(t) \qquad \xi^q \equiv \xi^q(q,t) \qquad \varphi = \varphi^{(0)}(q,t) + P\varphi^{(1)}(q,t). \tag{6}$$

Putting these forms into the last relation in (5), the terms independent of P take the form

$$\varphi^{(0)}_t - \beta^{(0)}(q)\varphi^{(0)} - \beta^{(1)}(q)\varphi^{(0)}_q - \beta^{(2)}(q)\varphi^{(0)}_{qq} = 0. \tag{7}$$

Equation (7) is of the same form as the original (3). There are in general an infinite number of linearly independent solutions and for each of these solutions we have a generator $\varphi^{(0)}(q,t)\partial/\partial P$. These generators arise because a homogeneous partial differential equation admits an infinite group of trivial symmetries corresponding to the fact that its solutions form a vector space. This infinite-parameter group forms an Abelian invariant subgroup of the maximal symmetry group. We shall be interested in the corresponding factor group and shall call this factor group the symmetry group G of equation (3).

The other relations in (5) will give us

$$\begin{aligned} \xi^q &= [A(t)+\tfrac{1}{2}B(q)\xi'_t]/B(q)' \\ \varphi^{(1)} &= C(t)-\tfrac{1}{2}A(t)\Psi(q)-\tfrac{1}{2}A_t(t)B(q)-\tfrac{1}{4}\xi'_tB(q)\Psi(q)-\tfrac{1}{8}\xi'_{tt}[B(q)]^2 \end{aligned} \tag{8}$$

where

$$B(q)=\int \mathrm{d}q/\sqrt{\beta^{(2)}(q)} \qquad \Psi(q)=\beta^{(1)}(q)B(q)'-(1/B(q)')' \tag{9}$$

Here $A(t)$ is an arbitrary function of t and $C(t)$ satisfies

$$C_t(t)+\tfrac{1}{4}\xi'_{tt}=A(t)f_1(q)+\xi'_tf_2(q)+A_{tt}(t)f_3(q)+\xi'_{ttt}f_4(q) \tag{10}$$

where

$$\begin{aligned} f_1(q) &= \beta^{(0)}(q)'/B(q)'-\tfrac{1}{2}\beta^{(1)}(q)\Psi(q)'-\tfrac{1}{2}(1/B(q)')^2\Psi(q)'' \\ &= [\beta^{(0)}(q)-F(q)]'/B(q)' \\ f_2(q) &= \beta^{(0)}(q)+\tfrac{1}{2}B(q)\beta^{(0)}(q)'/B(q)'-\tfrac{1}{4}\beta^{(1)}(q)[B(q)\Psi(q)]'-\tfrac{1}{4}(1/B(q)')^2[B(q)\Psi(q)]'' \\ &= [B(q)^2(\beta^{(0)}(q)-F(q))]'/2B(q)B(q)' \\ f_3(q) &= \tfrac{1}{2}B(q) \qquad f_4(q)=\tfrac{1}{8}[B(q)]^2 \end{aligned}$$

and

$$F(q)=\tfrac{1}{2}\beta^{(1)}(q)+\tfrac{1}{4}[\beta^{(1)}(q)B(q)']^2+\beta^{(1)}[\ln B(q)']'-\tfrac{1}{2}[1/B(q)']''/B(q)'+\tfrac{1}{4}[(1/B(q)')']^2. \tag{11}$$

It should be noted here that $B(q)$ cannot be a constant since $\beta^{(2)}(q)\neq 0$.

If the 'transition probabilities' $\beta^{(i)}$ are known, the symmetry group of the corresponding Fokker–Planck-type equation can be obtained (Sastry and Dunn 1985, Sastry *et al* 1987, Cicogna and Vitali 1989) by standard procedures. We, on the other hand, shall investigate what are the possible symmetry groups of (3) and the relations between the $\beta^{(i)}$ for each of these symmetry groups.

3. Symmetry classes

Since the left-hand side of (10) is a function of t only the right-hand side, which formally involves functions of q, must be independent of q. If all $f_i(q)$ are functionally independent then $A(t)=A_{tt}(t)=\xi'_t=\xi'_{ttt}=0$, and the symmetry group G will have only two generators

$$X^t=\mathrm{i}\partial/\partial t \qquad X_S=P\partial/\partial P \tag{12}$$

where X^t is the time translation operator and X_S is the scaling operator for P.

We now investigate the cases where G has other non-trivial generators. Referring to (10), we find that this will happen (Cicogna and Vitali 1989) when there exists a linear relationship between different $f_i(q)$ and 1. In analysing these relationships, we first note that f_3, f_4 and 1 cannot be linearly dependent since $B(q)$ is not a constant. So the possible relationships can be divided into two groups: (i) f_1 and f_2 are related with or without 1, f_3 and f_4; (ii) f_1 and f_2 are separately related with the other functions, namely 1, f_3 and f_4. Each of these two separate relationships of f_1 and f_2 in (ii) can combine with each other. If the relationships are denoted by the symbol R, we can have only the following cases:

(I) $R(1,f_1,f_2,f_3,f_4)$ (II) $R(1,f_1,f_2,f_3)$ (III) $R(1,f_1,f_2,f_4)$

(IV) $R(1,f_1,f_2)$ (V) $R(f_1,f_2,f_3,f_4)$ (VI) $R(f_1,f_2,f_3)$

(VII) $R(f_1,f_2,f_4)$ (VIII) $R(f_1,f_2)$

(IX) $R(1,f_2,f_3,f_4)$	(a) $R(1,f_1,f_3,f_4)$
(X) $R(1,f_2,f_3)$	(b) $R(1,f_1,f_3)$
(XI) $R(1,f_2,f_4)$	(c) $R(1,f_1,f_4)$
(XII) $R(1,f_2)$	(d) $R(1,f_1)$
(XIII) $R(f_2,f_3,f_4)$	(e) $R(f_1,f_3,f_4)$
(XIV) $R(f_2,f_3)$	(f) $R(f_1,f_3)$
(XV) $R(f_2,f_4)$	(g) $R(f_1,f_4)$
(XVI) $R(f_2)$	(h) $R(f_1)$.

With each case (IX) to (XVI) on the left we can associate any of the relationships (a) to (h) on the right. We have explicitly considered all these possible sets of linear relationships and found that, except for the following four classes, the symmetry group G has the trivial form consisting of X^t and X_S. The ω and b_i appearing in the following are all constants.

Class (A).

$$f_2(q) = b_1 - b_3 f_3(q) + \omega^2 f_4(q)$$
$$f_1(q) = -\tfrac{1}{3}b_3 + \tfrac{1}{4}\omega^2 f_3(q) \qquad \text{with } \omega \neq 0$$

so that

$$\beta^{(0)}(q) = b^1 - \tfrac{1}{3}b_3 B(q) + \tfrac{1}{16}\omega^2 [B(q)]^2 + F(q). \tag{13}$$

The symmetry group consists of the following six generators:

$$X^t = \mathrm{i}\partial/\partial t \qquad X_S = P\partial/\partial P$$
$$\left.\begin{aligned} Y_{(\pm)} &= \mathrm{e}^{\pm \mathrm{i}\omega t/2}\{(1/B(q)'(-\mathrm{i}\partial/\partial q) + [\pm(2b_3/3\omega - \tfrac{1}{4}\omega B(q) + \tfrac{1}{2}\mathrm{i}\Psi(q)]X_S\} \\ X_{(\pm)} &= \mathrm{e}^{\pm \mathrm{i}\omega t}\{X^t \mp (1/B(q)')[4b_3/3\omega - \tfrac{1}{2}\omega B(q)](-\mathrm{i}\partial/\partial q) \\ &\quad + [(\mathrm{i}b_1 + 4\mathrm{i}b_3^2/9\omega^2 \pm \tfrac{1}{4}\omega) \mp (2b_3/3\omega - \tfrac{1}{4}\omega B(q))\Psi(q) \\ &\quad - \tfrac{2}{3}\mathrm{i}b_3 B(q) + \tfrac{1}{8}\mathrm{i}(\omega B(q))^2]X_S\} \end{aligned}\right\} \tag{14a}$$

with the commutators

$$\left.\begin{aligned} &[X^t, Y_{(\pm)}] = \mp\tfrac{1}{2}\omega Y_{(\pm)} && [X^t, X_{(\pm)}] = \mp\omega X_{(\pm)} \\ &[Y_{(+)}, Y_{(-)}] = -\tfrac{1}{2}\mathrm{i}\omega X_S && [Y_{(+)}, X_{(-)}] = \omega Y_{(-)} \\ &[Y_{(-)}, X_{(+)}] = -\omega Y_{(+)} && [X_{(+)}, X_{(-)}] = 2\omega X^t + 2\mathrm{i}\omega(b_1 - b_3^2/\omega^2)X_S. \end{aligned}\right\} \tag{14b}$$

Class (B).

$$f_2(q) = b_1 - b_2 f_1(q) - b_3 f_3(q) + \omega^2 f_4(q)$$

so that

$$\beta^{(0)}(q) = [b_1 - \tfrac{1}{2}b_2^2] + b_0/[2b_2 + B(q)]^2 + \tfrac{1}{16}\omega^2[2b_2 + B(q)]^2 + F(q) \qquad \text{with } b_0, \omega \neq 0. \tag{15}$$

The symmetry group G consists of four generators

$$X^t = \mathrm{i}\partial/\partial t \qquad X_S = P\partial/\partial P$$

$$\begin{aligned} X_{(\pm)} = \mathrm{e}^{\pm\mathrm{i}\omega t}\{X^t &\mp (\mathrm{i}\omega/2B(q)')[2b_2 + B(q)](-\mathrm{i}\partial/\partial q) \\ &+ [\tfrac{1}{4}(4\mathrm{i}b_1 \pm \omega) \pm \tfrac{1}{4}\omega(2b_2 + B(q))\Psi(q) \\ &+ \tfrac{1}{2}\mathrm{i}b_2\omega^2 B(q) + \tfrac{1}{8}\mathrm{i}(\omega B(q))^2]X_S\} \end{aligned} \tag{16a}$$

with the commutators

$$[X^t, X_{(\pm)}] = \mp\omega X_{(\pm)} \qquad [X_{(+)}, X_{(-)}] = 2\omega X^t + 2\mathrm{i}\omega(b_1 - b_2^2/2\omega^2)X_S. \tag{16b}$$

Class (C).

$$f_2(q) = b_1 - b_3 f_3(q) \qquad f_1(q) = -\tfrac{1}{3}b_3 \qquad \text{for all } b_1, b_3$$

so that

$$\beta^{(0)}(q) = b_1 - \tfrac{1}{3}b_3 B(q) + F(q). \tag{17}$$

The symmetry group G consists of six generators

$$X^t = \mathrm{i}\partial/\partial t \qquad X_S = P\partial/\partial P$$

$$\begin{aligned} X_1 = tX^t &- (1/2B(q)')[b_3 t^2 + B(q)](-\mathrm{i}\partial/\partial q) \\ &+ \mathrm{i}[(b_1 t - \tfrac{1}{18}b_3^2 t^3) - \tfrac{1}{4}(b_3 t^2 + B(q))\Psi(q) - \tfrac{1}{2}b_3 tB(q)]X_S \\ X_2 = \tfrac{1}{2}t^2 X^t &- (1/2B(q)')[\tfrac{1}{3}b_3 t^3 + tB(q)](-\mathrm{i}\partial/\partial q) \\ &+ \mathrm{i}[(-\tfrac{1}{4}t + \tfrac{1}{2}b_1 t^2 - \tfrac{1}{72}b_3^2 t^4) - \tfrac{1}{4}(\tfrac{1}{3}b_3 t^3 + tB(q))\Psi(q) \\ &- \tfrac{1}{4}b_3 t^2 B(q) - \tfrac{1}{8}(B(q))^2]X_S \\ X_3 = (1/B(q)')&(-\mathrm{i}\partial/\partial q) + \mathrm{i}[\tfrac{1}{3}b_3 t + \tfrac{1}{2}\Psi(q)]X_S \\ X_4 = (t/B(q)')&(-\mathrm{i}\partial/\partial q) + \tfrac{1}{2}\mathrm{i}[\tfrac{1}{3}b_3 t^2 + t\Psi(q) + B(q)]X_S \end{aligned} \tag{18a}$$

with commutators

$$\begin{aligned} &[X^t, X_1] = \mathrm{i}X^t - \mathrm{i}b_3 X_4 - b_1 X_S \qquad [X^t, X_2] = \mathrm{i}X_1 + \tfrac{1}{4}X_S \\ &[X^t, X_3] = -\tfrac{1}{3}b_3 X_S \qquad [X^t, X_4] = \mathrm{i}X_3 \qquad [X_1, X_2] = \mathrm{i}X_2 \\ &[X_1, X_3] = -\tfrac{1}{2}\mathrm{i}X_3 \qquad [X_1, X_4] = \tfrac{1}{2}\mathrm{i}X_4 \qquad [X_2, X_3] = -\tfrac{1}{2}\mathrm{i}X_4 \\ &[X_3, X_4] = \tfrac{1}{2}X_S. \end{aligned} \tag{18b}$$

Class (D).

$$f_2(q) = b_1 - b_2 f_1(q) \qquad \text{for all } b_1, b_2$$

so that

$$\beta^{(0)}(q) = b_1 + b_0/[2b_2 + B(q)]^2 + F(q) \qquad \text{with } b_0 \neq 0. \tag{19}$$

The symmetry group G consists of four generators

$$X^t = \mathrm{i}\partial/\partial t \qquad X_S = P\partial/\partial P$$
$$X_1 = tX^t - (1/2B(q)')[2b_2 + B(q)](-\mathrm{i}\partial/\partial q) + \mathrm{i}[b_1 t - \tfrac{1}{4}(2b_2 + B(q))\Psi(q)]X_S$$
$$X_2 = \tfrac{1}{2}t^2 X^t - (t/2B(q)')[2b_2 + B(q)](-\mathrm{i}\partial/\partial q)$$
$$- \tfrac{1}{2}\mathrm{i}[(t/2 - b_1 t^2) + \tfrac{1}{2}t(2b_2 + B(q))\Psi(q) + b_2 B(q) + \tfrac{1}{4}(B(q))^2]X_S \tag{20a}$$

with commutators

$$[X^t, X_1] = \mathrm{i}X^t - b_1 X_S \qquad [X^t, X_2] = \mathrm{i}X_1 + \tfrac{1}{4}X_S$$
$$[X_1, X_2] = \mathrm{i}X_2 + \tfrac{1}{2}b_2^2 X_S. \tag{20b}$$

In the appendix we shall show how the class (A) was obtained. The heat equation in Sastry *et al* (1985) belongs to our class (C), the genetic equation belongs to our class (B) while the plasma physics equation belongs to our class (D). The two cases considered by Cicogna and Vitali (1989) belong to our classes (A) and (C) respectively.

4. Complete invariant bases

In this section we obtain the complete invariant bases in terms of which all invariants of the particular symmetry group can be functionally expressed. The method (Goursat 1945, Lie 1874, Rudra 1987) consists of writing the generators X_i of the group in the differential form $X_i \to \chi_i = \sum_{jk} C_{ij}^k x_k \partial/\partial x_j$, with the commutation relation $[\chi_i, \chi_j] = \sum_k C_{ij}^k \chi_k$ having the same structure constants C_{ij}^k as those of X_i. If $I_\alpha(x)$, $\alpha = 1, \ldots, p$ are the complete set of integrals of the differential equations $\chi_i I_\alpha(x) = 0$ for all i, then any invariant $I(x)$ defined by $\chi_i I(x) = 0$ for all i is a function of $I_\alpha(x)$, $\alpha = 1, \ldots, p$. The I_α then form the complete functional base of invariants. We now replace x_i by X_i in I_α and symmetrise the expressions, and obtain the invariant base of G. All the above four classes have two base invariants except when $b_3 \neq 0$ in class (C); one of these two invariants is always X_S. The complete sets for the four classes are as follows ($\mathcal{S}$ denotes symmetrisation).

Class (A).

$$I_S = X_S$$
$$I_t = 2[b_1 - b_3^2/\omega^2](X_S)^2 X^t + \mathcal{S}[-2X^t Y_{(+)} Y_{(-)} + X_{(+)}(Y_{(-)})^2 + X_{(-)}(Y_{(+)})^2]$$
$$-\mathrm{i}X_S[(X^t)^2 + (b_1 - b_3^2/\omega^2)(Y_{(+)}Y_{(-)} + Y_{(-)}Y_{(+)})$$
$$+ \tfrac{1}{2}(X_{(+)}X_{(-)} + X_{(-)}X_{(+)})]. \tag{21}$$

Class (B).

$$I_S = X_S$$
$$I_t = (b_1 - \tfrac{1}{2}b_2^2\omega^2)X_S X^t + \tfrac{1}{2}\mathrm{i}[X_{(+)}X_{(-)} + X_{(-)}X_{(+)} - (X^t)^2]. \tag{22}$$

Class (C).

(i) If $b_3 \neq 0$ then

$$I_S = X_S$$

(ii) if $b_3=0$ then

$$I_S = X_S$$

$$I_t = (X_S)^2[\tfrac{1}{4}X_1 + b_1X_2] - \tfrac{1}{2}iX_S[X^tX_2 + X_2X^t + X_3X_4 + X_4X_3 - (X_1)^2 + b_1(X_4)^2] + \mathscr{S}[X_1X_3X_4 - X_2(X_3)^2 - \tfrac{1}{2}X^t(X_4)^2]. \tag{23}$$

Class (D).

$$I_S = X_S$$

$$I_t = X_S[\tfrac{1}{4}X_1 + b_1X_2 - \tfrac{1}{2}b_2^2X^t] - \tfrac{1}{2}i[X^tX_2 + X_2X^t - (X_1)^2]. \tag{24}$$

5. Discussion

We discuss here a possible use of these symmetry groups of Fokker–Planck-type equations. One of the most important applications of these equations is in transport theories in either solid state devices or plasma physics. The 'transition probabilities' can be controlled by suitable doping of the devices or by appropriate external fields. By these means we can go from one to another of the classes (A), (B), (C) or (D). Among these transitions, those (i) from class (A) with $b_0=0$ to class (B) with $b_0\neq 0$, and (ii) from classes (C) with $b_0=0$ to class (D) with $b_0\neq 0$ satisfy group–subgroup relationships. On the other hand, the transitions (iii) from class (A) with $\omega\neq 0$ to class (C) with $\omega=0$, and (iv) from class (B) with $\omega\neq 0$ to class (D) with $\omega=0$ do not have any such group–subgroup relationship. If we look upon these transitions as phase transitions (Tolédano and Tolédano 1987) then cases (iii) and (iv) will be of first order without any order parameter; while the order of transitions in cases (i) and (ii) cannot be decided without further analysis, but there will be an order parameter.

Appendix

In this appendix we describe, as an example, the method of obtaining class (A), together with the symmetry group G of the Fokker–Planck-type equation. We first assume that (this is case IX together with case (a))

$$f_2(q) = b_1 - b_3f_3(q) + \omega^2 f_4(q) \qquad f_1(q) = k_1 - k_3^2f_3(q) - k_4^2f_4(q). \tag{A1}$$

Using (11) and (A1) we obtain

$$\begin{aligned}\beta^{(0)}(q) &= b_1 + b_0/(B(q))^2 - \tfrac{1}{3}b_3B(q) + \tfrac{1}{16}\omega^2(B(q))^2 + F(q)\\ \beta^{(0)}(q) &= k_0 + k_1B(q) - \tfrac{1}{4}k_3^2(B(q))^2 - \tfrac{1}{24}k_4^2(B(q))^3 + F(q).\end{aligned} \tag{A2}$$

In order that these two expressions for $\beta^{(0)}_{(q)}$ are simultaneously valid we get (remembering that $B(q)$ is not a constant)

$$k_4=0 \qquad k_3^2 = -\tfrac{1}{4}\omega^2 \qquad k_1 = -\tfrac{1}{3}b_3 \qquad k_0=b_1 \qquad b_0=0$$

and we obtain the form of $\beta^{(0)}(q)$ as in (13). Putting (A1) into (10) and equating the coefficients of $f_3(q)$ and $f_4(q)$ on the right-hand side separately to zero, we have

$$\xi^t_{ttt} + \omega^2\xi^t_t = 0 \qquad A_{tt}(t) + \tfrac{1}{4}\omega^2A(t) = b_3\xi^t_t. \tag{A3}$$

These have the solutions

$$\xi' = d_0 + [d_{(+)}\,e^{i\omega t} + d_{(-)}\,e^{-i\omega t}]$$
$$A(t) = [a_{(+)}\,e^{i\omega t/2} + a_{(-)}\,e^{-i\omega t/2}] - (\tfrac{4}{3}ib_3\omega)[d_{(+)}\,e^{i\omega t} + d_{(-)}\,e^{-i\omega t}] \tag{A4}$$
$$C_t(t) = -\tfrac{1}{4}\xi'_{tt} + b_1\xi'_t - \tfrac{1}{3}b_3A(t).$$

Putting these expressions into those of the velocity vectors in (8), we get the generators given in (14a).

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